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A NOTE ON THE PRABIR ROY SPACE

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A modification is given of the Prabir Roy construction of a zero-dimensional complete metric space which is not strongly zero-dimensional and which has weight $\aleph_1$. A generalisation is offered to encompass a similar recent construction due to Kulesza.

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complete metric space	large trees
inductive dimension	weight

1. Introduction

Let ind and Ind respectively denote the dimension functions in the small and in the large which are defined by an inductive reference to the dimension of the boundary of arbitrarily small neighbourhoods of, respectively, points or closed sets. These dimension functions coincide in separable metric spaces (see for example [2, p. 65]). In the nonseparable case the only known metric example, due to Roy [7, 8] or [6] has a discrepancy of 1 between the two functions. Roy's space has weight the continuum. Kunen asked whether in the absence of the continuum hypothesis the weight could still be $\aleph_1$.

The object of this note is to slightly modify Roy's construction (using "fast subsequences") in such a way that the weight of the space is lowered to $\aleph_1$. We thus prove:

Theorem 1.1. *There exists a complete metric space P with a basis of cardinality $\aleph_1$ such that*

$$\operatorname{ind} P = 0 \quad \textit{and} \quad \operatorname{Ind} P = 1.$$

An earlier draft of this paper claimed to use a weakened form of a combinatorial principle known to be independent of the continuum hypothesis; however, as the referee remarked, the weakened form is already (trivially) true in ZFC. In the meantime Kulesza [4] has given an alternative modification of the Prabir Roy construction again of weight $\aleph_1$ basing his modification on the "pressing down lemma". Both modifications simplify the detail of Roy's argument lending, one hopes, greater clarity to the original. Other modifications exist in the literature due to Mrówka [5] and Terasawa [10] but these have weight the continuum (even when squared).

The central combinatorial tool in Roy's construction, the indicator, which we modify in Section 4 to our "large tree", is a precursor of the independently and more recently discovered notion of "full set"; see [3]. At appropriate points our proof is therefore couched in the language of trees. We note that the variation of the Prabir Roy construction given by Terasawa [10] has the property (under CH) that the square of the space has the same dimensional discrepancy.

Before proceeding to the topological construction let us prepare some combinatorial tools. As usual we identify an ordinal α with its set of predecessors. For each $\alpha \in \omega_1$ we select a bijection, to be denoted for convenience $\underset{\sim}{\alpha}$, of the positive integers $\mathbb{N}$ onto α in such a way that the following property holds:

$$(\forall \beta \in \omega_1)(\forall \delta \in \omega_1)(\exists \alpha > \delta)$$
$$[\alpha \in \omega_1 \text{ and } \beta \text{ is a "fast subsequence" of } \underset{\sim}{\alpha}], \tag{C1}$$

where we define β to be a "fast subsequence" of $\underset{\sim}{\alpha}$ if there is a sequence of integers $\{n_i: i = 1, 2, \ldots\}$ such that

$$i < n_i, \quad \beta(i) = \underset{\sim}{\alpha}(n_i).$$

We note two obvious properties that are crucial:

$$\text{for any two uncountable sets } S_1, S_2 \subseteq \omega_1 \text{ there is a limit ordinal } \sigma \in \omega_1 \text{ with } |S_i \cap \sigma| = \omega \ (i = 1, 2); \tag{C2}$$

and

$$\text{for any } \beta \in \omega_1,\ \beta \text{ is a fast subsequence of } \omega_1\text{-many } \underset{\sim}{\alpha}. \tag{C3}$$

2. Construction of the space

Let I denote the set $\{-1, 1\} \times \omega_1$. We shall refer to the set $I_+ = \{1\} \times \omega_1$ as ω_1 and to the set $I_- = \{-1\} \times \omega_1$ as $-\omega_1$; for $\alpha \in \omega_1$, α will also ambiguously denote $\langle 1, \alpha \rangle$ while $-\alpha$ will denote $\langle -1, \alpha \rangle$. It is important that $\alpha \mapsto -\alpha$ is a bijection. We take the space I with discrete metric and form the product space $I^{\mathbb{N}}$ whose basic closed-and-open sets (*balls*) take the following form as in the classical definition of the Baire space $\mathbb{N}^{\mathbb{N}}$:

$$B(x) = \{z \in I^{\mathbb{N}}: z \text{ extends } x\},$$

where x is a finite sequence with terms in I. The Prabir Roy construction adds a *shell* of points $P(x)$ to every $B(x)$ where

$$P(x)=\{x\}\times\omega_1\times Z,$$

and Z is the set of all ω-sequences with terms in I_+. Letting X denote the set of all finite sequences x with terms in I we see that the underlying set in the Prabir Roy space is

$$P=I^{\mathbb{N}}\cup(X\times\omega_1\times Z).$$

A topology is introduced by specifying basic open sets (see [3, p. 39]). These are of two types. Type-1 sets take the form

$$\tilde{B}(x)=B(x)\cup\bigcup\{P(x')\colon x'=x \text{ or } x' \text{ extends } x\},$$

and contain the shell of $B(x)$ as well as the shells of all the sub-balls $B(x')$.

Before we can introduce Type-2 sets we recall that we have assumed fixed an bijective map $\underline{\alpha}:\omega\to\alpha$. Type-2 sets take the form (for $z_1,\ldots,z_{n-1}\in I_+$):

$$U(x,\alpha,z_1,\ldots,z_{n-1})=\bigcup_{i\geq n}\tilde{B}(x,\pm\underline{\alpha}(i),\mp\alpha,\mp z_1,\ldots,\mp z_{n-1})$$
$$\cup\{x\}\times\{\alpha\}\times Z(z_1,\ldots,z_{n-1}), \tag{1}$$

where

$$Z(z_1,\ldots,z_{n-1})=\{z\in Z\colon z\restriction n-1=(z_1,\ldots,z_{n-1})\}.$$

It is straightforward to check that these definitions validly introduce a topology in which the basic open sets are closed-and-open and it is evident that the induced topology on the subspace $I^{\mathbb{N}}$ is still the usual (product) topology on $I^{\mathbb{N}}$.

Let us make a number of remarks. Observe first the sign-flip in the definition of U and the sign-constancy thereafter.

We say that the *depth* of the ball $\tilde{B}(x)$ is m when the length of the sequence x is m. Note that the depth of the balls in formula (1) is $m+n+1$. In view of this we shall say that the *depth* of $U(x,\alpha,z_1,\ldots,z_{n-1})$ is $m+n+1$.

The basic open neighbourhoods $U(x,\alpha,z_1,\ldots,z_{n-1})$ of the point $\xi=(x,\alpha,z)$ may be seen as built from basic open neighbourhoods of the points:

$$\xi_i^\pm=(x,\pm\alpha(i),\mp\alpha,\mp(z_1,z_2,\ldots,z_n,\ldots)),$$

which are "uniform" copies of (α,z) sited in $\tilde{B}(x,\pm\alpha(i))$.

Remark 2.1. If $\tilde{B}(x)\cap\tilde{B}(x')\neq\emptyset$, then also $B(x)\cap B(x')\neq\emptyset$.

Remark 2.2. If $U(x,\alpha,z_1,\ldots,z_{n-1})\cap U(x',\alpha',z_1',\ldots,z_{m-1}')\neq\emptyset$, then for some i, j and $\varepsilon,\delta\in\{\pm1\}$

$$B(x,\varepsilon\underline{\alpha}(i),-\varepsilon\alpha,-\varepsilon z_1,\ldots)\cap B(x',\delta\underline{\alpha}'(j),-\delta\alpha',-\delta z_1',\ldots)\neq\emptyset,$$

i.e., intersection cannot occur only on the shell.

Notation. $x < x'$ means that x' properly extends x. $x^\frown x'$ or (x, x') denotes the concatenation of the sequences x and x'. For $x \in X$, we use $|x|$ to denote the length (possibly 0) of the sequence x.

3. Proof of metrizability

This proceeds along the same lines as Roy's proof. We invoke Moore's metrization theorem which requires the existence of a family of open covers $\mathcal{G}_n$ $(n = 1, 2, \ldots)$ which are star-refining, viz for every point p and every open neighbourhood U of p there exists an open neighbourhood V of the point p and an integer N so that

$$\mathrm{St}(V, \mathcal{G}_N) \subseteq U,$$

where St denotes the star of V in $\mathcal{G}_N$ (the union of all sets of $\mathcal{G}_N$ meeting V). See [1, p. 409] or [6, p. 80].

In the current situation take $\mathcal{G}_n$ to consist of those sets $\tilde{B}(x)$ and those sets $U(x, \alpha, z_1, \ldots, z_{m-1})$ whose depth is n or more. Metrizability follows from the next two lemmas.

Lemma A ("Flip-over Lemma"). *For $(x, \alpha, z) \in X \times \omega_1 \times Z$ and given n let $N = |x| + n + 1$. If $W \in \mathcal{G}_N$ and $W \cap U(x, \alpha, z_1, \ldots, z_{n-1}) \neq \emptyset$, then*

(i) $W = \tilde{B}(x')$ for some $x' \in X$ implies $W \subseteq U(x, \alpha, z_1, \ldots, z_{n-1})$,

(ii) $W = U(x', \alpha', z'_1, \ldots, z'_{k-1})$ for appropriate $x', \alpha', z'_1, \ldots, z'_{k-1}$ implies that either $|x'| = N - 1$, in which case $W \subseteq \tilde{B}(x') \subseteq U(x, \alpha, z_1, \ldots, z_{n-2})$ provided $n \geq 2$, or $W \subseteq U(x, \alpha, z_1, \ldots, z_{n-1})$.

Proof. Suppose first $W = \tilde{B}(x')$ for some $x' \in X$. Thus $|x'| > |x|$. It follows that for some i and some sign $\pm$

$$\tilde{B}(x') \cap \tilde{B}(x, \pm\underline{\alpha}(i), \mp\alpha, \mp z_1, \ldots, \mp z_{n-1}) \neq \emptyset.$$

By Remark 2.1

$$(x') \cap B(x, \pm\underline{\alpha}(i), \mp\alpha, \mp z_1, \ldots, \mp z_{n-1}) \neq \emptyset,$$

and by reference to an element in the intersection it may be seen that x' extends the sequence

$$(x, \pm\underline{\alpha}(i), \mp\alpha, \mp z_1, \ldots, \mp z_{n-1})$$

(as $|x'| \geq |x| + n + 1$).

We now turn to case (ii). Suppose then that for some $x' \in X$, $\alpha \in \omega_1$ and $z'_1, \ldots, z'_{k-1}$ we have

$$W = U(x', \alpha', z'_1, \ldots, z'_{k-1}).$$

Again by Remarks 2.1 and 2.2 we need only consider the situation where, for some i and j, and, some sign choices ε and δ, we have

$$B(x', \varepsilon\alpha'(j), \ldots) \cap B(x, \delta\alpha(i), \ldots) \neq \emptyset. \tag{2}$$

We draw inferences from the various possible cases from Fig. 1. Figure 1 indicates blocks of terms of arbitrary sign with a wavy line, blocks of terms of constant sign with a shaded box and the term of opposite sign which precedes the box with a circle.

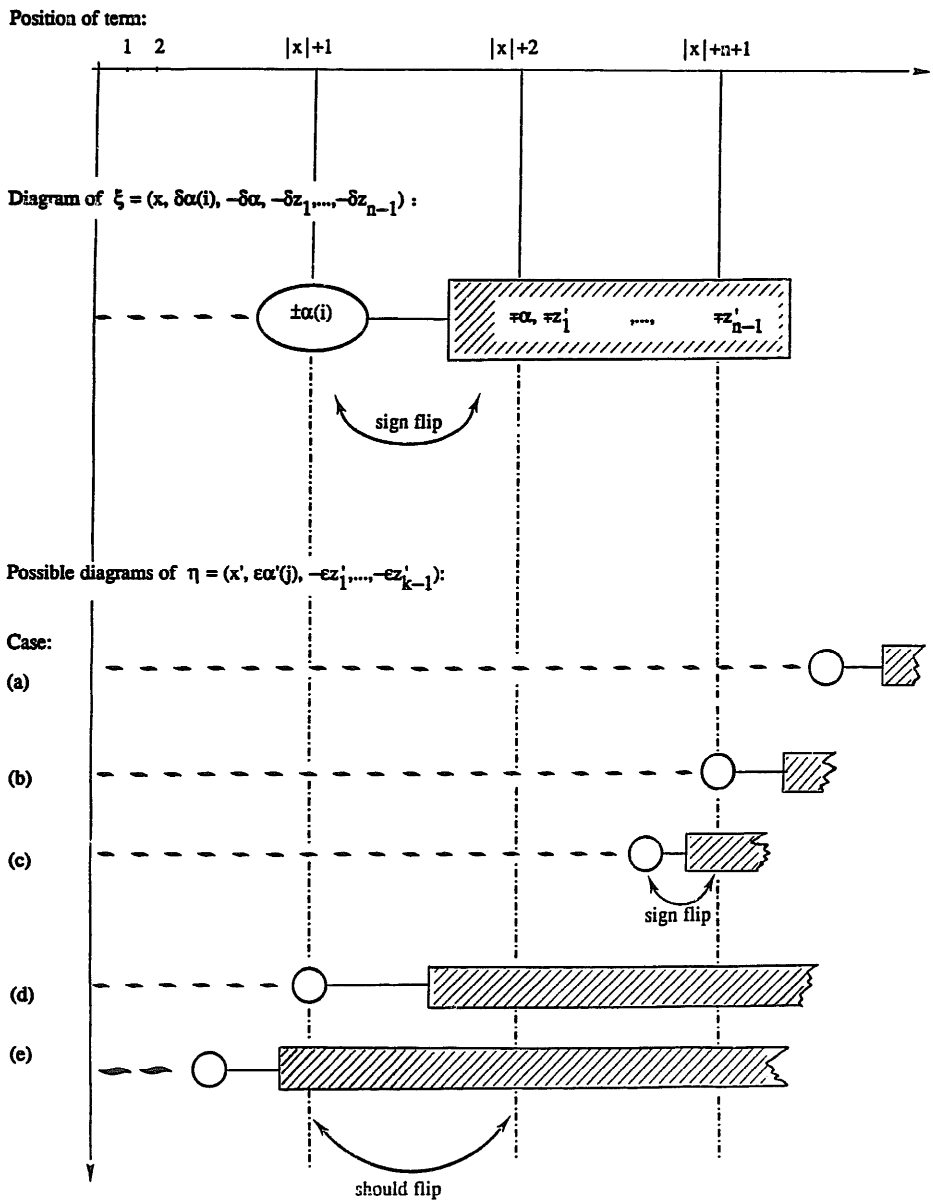

Fig. 1

Let $\beta \in I^{\mathbb{N}}$ belong to the intersection in (2). Since β must agree on its first $|x|+n+1$ terms with both sequences $\xi=(x, \delta\alpha(i), \dots)$ and $\eta=(x', \varepsilon\alpha'(j), \dots)$ we conclude as follows (see Fig. 1):

Case (a) here $B(x') \subseteq B(x, \delta\alpha(i), \dots)$;

Case (b) see below;

Case (c) a contradiction occurs between constancy of sign in ξ and the sign-flip in η, unless $n=1$, whereupon $|x|+2=|x'|+1$ and Case (b) arises;

Case (d) $\delta\alpha=\varepsilon\alpha'$ and so either the equality or inclusion $W \subseteq U(x, \alpha, z_1, \dots, z_{n-1})$ follows;

Case (e) a contradiction occurs between constancy of sign in η and the sign-flip in ξ.

Noninclusion occurs in Case (b), but here we notice that

$$|x'|+1=|x|+n+1,$$

i.e., $|x'|=N-1$, as asserted. □

Lemma B. *For any $x \in X$ there is N so large that if*

$$W \in \mathscr{G}_N \quad \textit{and} \quad W \cap \tilde{B}(x) \neq \emptyset,$$

then

(i) $W=\tilde{B}(x')$ *implies* $W \subseteq \tilde{B}(x)$.

(ii) $W=U(x'; \alpha'; z'_1, \dots, z'_{k-1})$ *implies either* $W \subseteq \tilde{B}(x)$, *or* $|x'|=|x|+1$, *in which case* $W \subseteq \tilde{B}(x') \subseteq \tilde{B}(x_1, \dots, x_{|x|-1})$ *provided* $|x| \geq 1$.

Proof. In view of Remark 2.1, case (i) will evidently hold if we arrange for $N \geq |x|$. We compute how large N must be for (ii) to hold. Assuming that $W=U(x', \alpha', z'_1, \dots, z'_{k-1})$ and $W \cap \tilde{B}(x) \neq \emptyset$, there exists $i \geq N$ and a choice of sign $\varepsilon \in \{\pm 1\}$ such that

$$\tilde{B}(x) \cap \tilde{B}(x', \varepsilon\alpha'(i), -\varepsilon\alpha', -\varepsilon z'_1, \dots, -\varepsilon z'_{k-1}) \neq \emptyset. \tag{3}$$

Let us write

$$\xi=(x', \varepsilon\alpha'(i), -\varepsilon\alpha'). \tag{4}$$

First suppose that

$$\xi \text{ is an initial segment of } x.$$

Thus for some $l \leq |x|$ we have

$$\alpha'=|x_l|, \qquad \underset{\sim}{\alpha}'(i)=|x_{l-1}|. \tag{5}$$

Since the function $\underset{\sim}{\alpha}: \omega \to \omega_1$ is injective we can compute from x_h, x_{h-1} for each $h \leq |x|$ the unique integer m_h such that

$$|\underset{\sim}{x_h}|(m_h)=|x_{h-1}|, \tag{6}$$

where $|x_h|$ and $|x_{h-1}|$ refer to the unsigned ordinals corresponding to x_h and x_{h-1}. Thus if $N = N(x)$ is selected so that

$$N > \max\{m_h\colon 1 < h \leqslant |x|\}, \tag{7}$$

equation (6) will contradict equation (5) since $i \geqslant N$. The condition on N just derived thus ensures that ξ cannot be an initial segment of x.

Next suppose that

$$x \text{ is an initial segment of } \xi. \tag{8}$$

Evidently if $|x| \leqslant |x'|$, the conclusion $W \subseteq \tilde{B}(x)$ is trivial. By (4) this leaves open the possibilities that

$$|x| = |x'| + 1 \quad \text{or} \quad |x| = |x'| + 2.$$

In view of (3) the latter equality says that x and ξ are equal and this has already been precluded by the argument leading to condition (7) which we have placed on N. The former case evidently gives $x' < x$, so

$$W \subseteq \tilde{B}(x_1, \ldots, x_{|x|-1}),$$

where $|x|$ is the length of the sequence x, assumed positive.

Thus (i) and (ii) hold if we take

$$N = |x| + \max\{m_h\colon 1 < h \leqslant |x|\}. \quad \square$$

Metrizability now follows easily. Let p be given and let U be any basic open neighbourhood of p. First suppose $p = (x, \alpha, z)$ for some x, α, z and say

$$U = U(x, \alpha, z_1, \ldots, z_{n-1}).$$

Let

$$V = U(x, \alpha, z_1, \ldots, z_{n-1}, z_n)$$

and

$$M = |x| + n + 2.$$

By Lemma A if $W \cap V \neq \emptyset$ for $W \in \mathscr{G}_M$, then either $W \subseteq V$ or $W = U(x', \alpha', z'_1, \ldots, z'_{k-1})$ where $|x'| = M - 1$. It follows that

$$\tilde{B}(x') \subseteq U(x, \alpha, z_1, \ldots, z_{n-1})$$

(see case (ii) in Lemma A) and so

$$\mathrm{St}(V, \mathscr{G}_M) \subseteq U.$$

Now suppose $p \in I^{\mathbb{N}}$ and that $U = \tilde{B}(\hat{x})$ for some $\hat{x} \in X$. Thus $\hat{x} = p \restriction m$ for some m. Let $x = p \restriction m+1$ and $V = \tilde{B}(x)$. Choose N as in Lemma A. Thus if $W \in \mathscr{G}_N$ and $W \cap \tilde{B}(x) \neq \emptyset$, we have that either $W \subseteq \tilde{B}(x)$ or $W = U(x', \alpha', z'_1, \ldots, z'_{k-1})$ for some x', α', z' and

$$W \subseteq \tilde{B}(x).$$

Thus again we have

$$\mathrm{St}(V, \mathcal{G}_N) \subseteq U.$$

Evidently the weight of the metric space is $\aleph_1$ (counting up the basic open sets). It is clear also that the space P is complete with respect to the sequence $\langle \mathcal{G}_n : n = 1, 2, \ldots \rangle$ of open covers.

Lemma C. *Let $\mathcal{H}_m$ consist of the basic open sets of depth m; then no point belongs to more than two elements of $\mathcal{H}_m$.*

Proof. If x and x' are distinct sequences of equal length m, then $\tilde{B}(x) \cap \tilde{B}(x') = \emptyset$. Next suppose

$$U(x, \alpha, z_1, \ldots, z_{n-1}) \cap U(x', \alpha', z'_1, \ldots, z'_{k-1}) \neq \emptyset,$$

and $m = |x| + n + 1 = |x'| + k + 1$. Thus for some natural numbers i and j and for some ε, δ in $\{\pm 1\}$:

$$B(x, \varepsilon\alpha(i), -\varepsilon\alpha, -\varepsilon z_1, \ldots, -\varepsilon z_{n-1})$$
$$\cap\, B(x', \delta\alpha'(j), -\delta\alpha'(j), -\delta\alpha', -\delta z'_1, \ldots, -\delta z'_{k-1}) \neq \emptyset.$$

Since the indexing sequences are of equal length and both contain sign-flips it follows (by working backwards along the indexing sequences) that the sign-flips have to occur at the same places and hence that $n = k$ and $x = x'$ and $\alpha = \alpha'$.

Evidently if $m = |x'| = |x| + n + 1$ and

$$U(x, \alpha, z_1, \ldots, z_{n-1}) \cap \tilde{B}(x') \neq \emptyset,$$

then $x' = (x, \varepsilon\alpha(i), -\varepsilon\alpha, -\varepsilon z_1, \ldots, -\varepsilon z_{n-1})$ for some natural number i and ε in $\{\pm 1\}$. The lemma is now clear. □

Corollary. $\mathrm{Ind}(P) \leq 1$.

Proof. By Lemma C, $\dim(P) \leq 1$ (see for example [7, p. 251]). Hence, since $\dim = \mathrm{Ind}$ (in a metric space), the corollary follows. □

4. Proof that Ind = 1

By a *large tree* (of *height* n) we shall mean a tree of sequences taken from $\omega_1^{\leq n}$ such that all but the terminal nodes have valency ω_1 (i.e., have ω_1 immediate successors). Notions such as these are also studied in [9]. Sequences of length n in such a tree are called branches. The empty sequence $\langle\,\rangle$ is called the *root* of the tree. We write Π_n for ω_1^n. Let $\mathcal{H}$ be a family of basic open sets of any type.

Lemma 4.1. *Let $x \in X$, $\alpha \in \omega_1$, $\pi \in \Pi_{n-1}$ satisfy:*

$$(\forall \beta \in \omega_1)(\exists \text{ a large tree } K_\beta)(\forall \text{ branches } k \in K_\beta)$$
$$[U(x, \alpha, \pi^\frown\langle\beta\rangle^\frown k) \in \mathscr{H}].$$

Then there exists a large tree K such that for all branches k in K

$$U(x, \alpha, \pi^\frown k) \in \mathscr{H}.$$

Proof. Put $T_m = \{\beta \in \omega_1 \colon K_\beta \text{ has height } m\}$. Then, for some m, $|T_m| = \omega_1$. The desired tree is then

$$K = \{\langle\beta\rangle^\frown k \colon \beta \in T_m \text{ and } k \in K_\beta\}. \qquad \square$$

Remark. Lemmas 4.1–4.3 are true if "large" tree is redefined so that all but the terminal nodes of the tree are of valency ω. In this case we obtain Roy's notion of an "indicator". The proof of the Main Lemma is valid for either notion. Our point of view is dictated by our earlier peripeteia with set-theoretic hypotheses.

Lemma 4.2. *Suppose basic open subsets of members of $\mathscr{H}$ are also in $\mathscr{H}$ and that $\{x\} \times \{\alpha\} \times Z \subseteq \bigcup \mathscr{H}$. Then there is a large tree K so that*

$$U(x, \alpha, \pi) \in \mathscr{H},$$

for all branches π of K.

Proof. Suppose no such tree exists, then by Lemma 4.1 there exists β_1 so that for no large tree K_{β_1}, is it true that

$$U(x, \alpha, \langle\beta_1\rangle^\frown k) \in \mathscr{H}$$

for all branches k of K_{β_1}. Applying Lemma 4.1 again we deduce that there is β_2 so that for no large tree K_{β_2} is it true that for all branches k of K_{β_2}

$$U(x, \alpha, \langle\beta_1\rangle^\frown\langle\beta_2\rangle^\frown k) \in \mathscr{H}.$$

Continuing inductively we build up a sequence

$$z = \langle\beta_1, \beta_2, \ldots\rangle \in \omega_1^{\mathbb{N}}.$$

Hence for some n and some $H \in \mathscr{H}$,

$$(x, \alpha, z) \in U(x, \alpha, \beta_1, \ldots, \beta_{n-1}) \subseteq H.$$

Consequently for every $k \in \Pi_1$ we have

$$U(x, \alpha, \langle\beta_1, \ldots, \beta_{n-1}\rangle^\frown k) \in \mathscr{H}.$$

This is a contradiction. $\square$

Lemma 4.3. *Let U_1, U_2 be open and suppose that*

$$\{x\} \times \{\alpha\} \times Z \subseteq U_1 \cup U_2.$$

Then, there exists $j \in \{1, 2\}$ and a large tree K such that for all branches k of K

$$U(x, \alpha, k) \subseteq U_j.$$

Proof. Let $\mathscr{H}$ consists of basic open sets refining either of U_1 or U_2. Apply Lemma 4.2 to obtain a large tree K_0 so that for all branches π of K_0

$$U(x, \alpha, \pi) \in \mathscr{H}.$$

Label the terminal node of π with a 1 or 2 according as $U(x, \alpha, \pi)$ refines U_1 or U_2. Then there exists a large tree K which is a subtree of K_0 with constant labelling of its terminal nodes. [To see this label the predecessors of the terminal nodes with a 1 (or 2) according as the node has ω_1 successors labelled with 1 (or respectively 2). Repeating this process by backward induction yields a label $j = 1$ or 2 at the root of the tree K_0. Now prune K_0 so as to leave only nodes labelled j.] $\square$

Main Lemma. *Suppose $P = U_1 \cup U_2$ and U_1, U_2 are open. Suppose also that $x \in X$ is such that there are large trees K_1 and K_2 such that for all branches k_1 of K_1 and k_2 of K_2*

$$\tilde{B}(x^\frown k_1) \subseteq U_1, \qquad \tilde{B}(x^\frown(-k_2)) \subseteq U_2.$$

Then there exists a proper extension x' of x and two large trees L_1 and L_2 so that for all branches l_1 of L_1 and l_2 of L_2 we have

$$\tilde{B}(x'^\frown l_1) \subseteq U_1, \qquad \tilde{B}(x'^\frown(-l_2)) \subseteq U_2.$$

Proof. For $i = 1, 2$ let S_i consist of the nodes of height 1 of K_i. By assumption $S_i \subseteq \omega_1$ and S_i is uncountable.

Hence, by property (C2), there is a limit ordinal σ such that for $i = 1, 2$

$$|\sigma \cap S_i| = \omega.$$

Pick two disjoint sequences of integers $1 < m_1 < m_2 < \cdots$ and $1 < n_1 < n_2 < \cdots$ so that for all j

$$\underline{\sigma}(m_j) \in S_1 \quad \text{and} \quad j < m_j,$$

$$\underline{\sigma}(n_j) \in S_2 \quad \text{and} \quad j < n_j.$$

Finally, let $S = \{\alpha \in \omega_1 : \underline{\sigma} \text{ is a fast subsequence of } \underline{\alpha}\}$. Using Lemma 4.3 we may choose for each $\alpha \in S$ an integer $j(\alpha) \in \{1, 2\}$ and a large tree K_α so that for all branches k of K_α we have

$$U(x, \alpha, k) \subseteq U_{j(\alpha)}.$$

For $j = 1, 2$ let $S(j) = \{\alpha \in S : j(\alpha) = j\}$. Since S is uncountable one of $S(1)$, $S(2)$ is uncountable. Say it is $S(1)$. Put

$$T_m = \{\alpha \in S(1) : K_\alpha \text{ has height } m\}.$$

Now pick N so that T_N is uncountable. Our objective is now to obtain L_2 from K_2 and to construct L_1 from the trees $\{K_\alpha : \alpha \in T_N\}$.

To obtain L_2 we put $c = \underline{\sigma}(n_N)$ and $x' = x^\frown\langle -c\rangle$. Let h be the height of K_2; we put

$$L_2 = \{\langle \rho(2), \ldots, \rho(h), \gamma\rangle : \rho \in K_2 \text{ and } \rho(1) = c \text{ and } \gamma \in \omega_1\};$$

$$L_1 = \{\langle \alpha\rangle^\frown \rho : \alpha \in T_N \text{ and } \rho \in K_\alpha\}.$$

Observe that $c \in S_2$ so c is a node of level 1 in K_2 hence L_2 is a large tree. Also since T_N is uncountable L_1 is a large tree. Note that if $\alpha \in T_N$, then for some integer p, $\underline{\sigma}(n_N) = \underline{\alpha}(p)$. Evidently, since $\underline{\sigma}$ is a fast subsequence of $\underline{\alpha}$, $p \geq n_N > N$. Now if k is a branch of K_α, k has length N. Thus

$$\tilde{B}(x, -c, \alpha, k) = \tilde{B}(x, -\underline{\alpha}(p), \alpha, k) \subseteq U(x, \alpha, k) \subseteq U_1,$$

since $p \geq N+1$. Thus L_1 fulfills the required role.

Now if l is a branch of L_2, then for some $\rho \in K_2$ with $\rho(1) = c$, we have $l = \langle \rho(2), \ldots, \rho(h), \gamma \rangle$, so

$$\tilde{B}(x, -c, -\rho(2), \ldots, -\rho(h), -\gamma) = \tilde{B}(x, -\rho, -\gamma) \subseteq \tilde{B}(x, -\rho) \subseteq U_2.$$

Thus L_2 fulfills the role required of it.

This completes the proof of the Main Lemma. $\square$

Theorem 4.4. $\operatorname{Ind}(P) = 1$.

Proof. It suffices to prove $\operatorname{Ind}(P) \geq 1$. With this aim in mind let

$$C = \bigcup_{\rho \in \Pi_2} \tilde{B}(\langle\ \rangle, \rho) \quad \text{and} \quad A = \operatorname{cl} C,$$

$$D = \bigcup_{\rho \in \Pi_2} \tilde{B}(\langle\ \rangle, -\rho) \quad \text{and} \quad B = \operatorname{cl} D.$$

We delay checking that $A \cap B = \emptyset$. Now let U_1, U_2 be disjoint open sets with $A \subseteq U_1$, $B \subseteq U_2$ and $P = U_1 \cup U_2$. We observe that $K_1 = K_2 = \Pi_2$ obey the hypothesis of the Main Lemma.

We may thus obtain by an inductive application of the Main Lemma a sequence of extensions of $\langle\ \rangle = x_0$, say $x_1 < x_2 < x_3 < \cdots < x_n$ and large trees L_1^n, L_2^n (where $L_1^0 = K_1$ and $L_2^0 = K_2$) so that for each n

$$\tilde{B}(x_n \frown l_1) \subseteq U_1 \quad \text{and} \quad \tilde{B}(x_n \frown (-l_2)) \subseteq U_2$$

for all branches l_1 of L_1^n and l_2 of L_2^n. It follows that if $\xi \in I^{\mathbb{N}}$ is the common extension of $\langle x_n \colon n = 0, 1, 2, \ldots \rangle$, then $\xi \in \operatorname{cl} U_1 \cap \operatorname{cl} U_2$ and this contradicts the disjointness of U_1 and U_2.

Finally we check that $A \cap B = \emptyset$.

First consider any point ξ in $I^{\mathbb{N}}$. If $\tilde{B}(x)$ is an open neighbourhood of ξ of depth at least 2, then we have

$$A \cap \tilde{B}(x) \neq \emptyset \Rightarrow C \cap \tilde{B}(x) \neq \emptyset \Rightarrow x(1) > 0 \text{ and } x(2) > 0,$$

while, similarly,

$$B \cap \tilde{B}(x) \neq \emptyset \Rightarrow x(1) < 0 \text{ and } x(2) < 0.$$

Thus ξ cannot lie in $A \cap B$.

Now consider a point ξ in $P\backslash I^{\mathbb{N}}$. Thus for some $x \in X$, $\xi \in P(x)$. If $x = \langle\ \rangle$, then none of the balls

$$\tilde{B}(\langle\ \rangle, \varepsilon\underset{\sim}{\alpha}(i), -\varepsilon\alpha)$$

can meet C or D because of the sign-flip. Let $x = \langle x_1, \ldots, x_l \rangle$ where $l \geq 1$. Consider a neighbourhood W of ξ meeting A. Then

$$W \cap A \neq \emptyset \Rightarrow W \cap C \neq \emptyset.$$

If $l = 1$, then for some α, and for some i and some $\varepsilon \in \{\pm 1\}$, $\tilde{B}(x_1, \varepsilon\alpha(i), \varepsilon\alpha) \cap C \neq \emptyset$ so $x_1 > 0$ and $\varepsilon = 1$, whereupon

$$\tilde{B}(x_1, \pm\alpha(i), \mp\alpha) \cap D = \emptyset.$$

If $l \geq 2$ we obtain $x_1 > 0$ and $x_2 > 0$ if $W \cap C \neq \emptyset$, whereupon $W \cap D = \emptyset$.
This completes the proof. □

Post-script. It is natural to inquire into the relationship between the construction here and that due to Kulesza. The following generalisation comes to mind. For each limit ordinal α choose disjoint subsets $S_i(\alpha)$ of α and now define the Type-2 open sets as follows:

$$U(x, \alpha, z_1, \ldots, z_{n-1}) = \bigcup_{\gamma \in S_n(\alpha)} B(x, \pm\gamma, \mp\alpha, \mp z_1, \ldots, \mp z_{n-1})$$
$$\cup \{x\} \times \{\alpha\} \times Z(z, \ldots, z_{n-1}).$$

It may be verified that this definition continues to give rise to a metric space; only the definition of m_h in Lemma B is adjusted to be the m such that $|x_{h-1}| \in S_m(|x_h|)$. The choice $S_i(\alpha) = [\lambda(i), \lambda(i+1))$ then leads to a proof that the space has large inductive dimension equal to unity. We hope to develop this theme in a forthcoming note. On the other hand it is also possible to take the Kulesza approach and to topologise $\omega_1\backslash\omega$ so that for each α the set $\alpha\backslash F$ (where F is a finite set) is open and to repeat the Kulesza coding devices on $(\omega_1\backslash\omega)^{\mathbb{N}}$ to obtain his counterpart to the construction of the current paper.

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